with the ratio $d'd=0.016$. However, for the 2.54 cm test piece, the error is amplified slightly. This may be because a longer duration of time (25 sec) was required in the experiment, and additional error might result from any difficulty involved in maintaining a constant heat flux. The measurement of temperature response within a $\tau=0.1$ interval is difficult, since the range of temperature response in general is small and the accuracy of the recording device is about 0.01 mv which means an error in measured temperature of 0.3°F. In addition, since it required 0.3 sec to place the test piece into position, the data below $\tau=0.1$ will not be accurate enough for estimation of the temperature distortion. However, the data presented in Figs. 3 and 4, which are for $\tau=0.3$ and 0.5, can be used as a guide to correct the measured response of the thermocouple before it is used in the inversion, based on a one-dimensional theoretical model[1-6] for prediction of transient surface temperature and heat flux. Measurements at a time $\tau>0.5$ were essentially the same as those of steady-state heat conduction. Therefore, the correction of error arising from cavity disturbance on a one-dimensional model may be taken to be the same as that at $\tau=0.5$. The previously mentioned error measurements are qualitatively verified by Chen and Li's analysis,[10] in which the upper and lower bounds of possible errors are numerically simulated.

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A Numerical Approach to Ionized Nonequilibrium Boundary Layers

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Introduction

NONEQUILIBRIUM ionized flows are characterized by existence of electron thermal regions in which the electron temperature deviates from both ion and atom temperature. In a nonequilibrium ionized boundary-layer flow, the electron thermal layer along a wall surface often becomes much thicker than the viscous boundary layer because of high electron thermal conductivity. For the boundary layer with this type of two-layer structure, the conventional numerical approach leads to serious difficulty in the computational scheme. The present Note shows that it can be overcome by introducing such a transformation of independent variables that the semi-infinite flow region is projected onto a confined region.

We consider a nonequilibrium ionized boundary layer which is induced by a shock wave along the side of a shock tube. The analysis is made for a steady, two-dimensional flow of a nonequilibrium argon plasma behind a shock front, based on the assumptions of quasi-charge-neutrality, zero net current, same temperatures for atoms and ions, no external field, and no radiative effect. Applying these assumptions to three-fluid conservation equations[1] for atoms, singly charged ions and electrons, we obtain a system of simultaneous equations for the following variables[2]: the overall density ρ, the mean macroscopic velocity components u and v, the total enthalpy H, the atom temperature T_a, the electron temperature T_e, and the degree of ionization α. After normalizing the variables, the conservation equations of momentum, total energy, charged species, and electron energy are numerically solved by a finite-difference scheme, which is based on the method of quasilinearization.[3]

Basic Equations

We conveniently choose the coordinate system fixed with respect to the shock front. The x-axis is taken along the wall and the y-axis along the shock front. In this coordinate system, the boundary layer is steady so long as the shock speed is assumed to be constant.

We define the dimensionless coordinate (ξ, η) as

$$\xi=\frac{x}{U_0 t_s}, \quad \eta=\frac{U_0}{\{2U_0(\rho\mu)_0 x\}^{1/2}}\int_0 \rho dy' \tag{1}$$

where U_0 is the shock speed and $(\rho\mu)_0$ the product of the density and the viscosity of the gas ahead of the shock wave. In the following, the subscript 0 denotes the gas properties ahead of the shock wave. The time t_s may be chosen arbitrarily, and utilized as a time-scaling parameter in the numerical scheme.

The stream function ψ is transformed to a dimensionless function f as

$$\psi=\{2U_0(\rho\mu)_0 x\}^{1/2} f, \quad f_\eta=u/U_0 \tag{2}$$

The dimensionless variables F_j $(j=1,2,3,4)$ are defined as

$$F_1=\frac{U_0}{u^*(\xi)}f_\eta\text{-}1,\ F_2=\frac{H}{H_0}\text{-}1, F_3=\frac{\alpha}{\alpha^*(\xi)},\ F_4=\frac{T_e}{T_e{}^*\xi)} \tag{3}$$

where the superscript * denotes the flow properties in the inviscid, external free stream behind the shock front. The external flow properties can be obtained by solving a system of the equations for one-dimensional, inviscid flow of a nonequilibrium ionized gas.[4] With these variables, the conservation equations of momentum, energy, charged species, and electron energy can be written in familiar forms as

$$a_j[b_j(F_j)_\eta]_\eta + f(F_j)_\eta = 2\xi[f_\eta(F_j)_\xi - f_\xi(F_j)_\eta] + d_j, \qquad (j=1,2,3,4) \tag{4}$$

where

$$a_1=a_2=a_3=1, \qquad a_4=5/(3\alpha)$$

$$b_1=\rho\mu/(\rho\mu)_0, b_2=b_1/Pr, b_3=b_2 Le, b_4=b_2\lambda_e/\lambda$$

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$$d_1 = -2\xi\left(\frac{\rho_0}{\rho} - f_\eta^2\frac{U_0}{u^*}\right)\frac{U_0}{u^*}\frac{d}{d\xi}\left(\frac{u^*}{U_0}\right)$$

$$d_2 = -\left[(b_1-b_2)\frac{U_0^2}{H_0}f_\eta f_{\eta\eta} + b_2\left(1+\frac{5}{2}\frac{RT_0}{I}\theta\right)\frac{I}{H_0}\{(b_3-b_2)\alpha_\eta+\Gamma\} + (b_4-b_2\alpha)\frac{5}{2}\frac{RT_0}{H_0}\theta_\eta\right]_\eta$$

$$d_3 = -\frac{1}{\alpha^*}\left[\Gamma_\eta + 2\xi\dot{\alpha}t_s - 2\xi f_\eta\frac{\alpha}{\alpha^*}\frac{d\alpha^*}{d\xi}\right]$$

$$d_4 = -\frac{1}{\alpha}\frac{\theta_\eta}{\theta^*}(b_3\alpha_\eta+\Gamma) + \frac{2}{3}\frac{\theta}{\theta^*}\left[(f+2\xi f_\xi + b_3\frac{\alpha_\eta}{\alpha}+\frac{\Gamma}{\alpha})(\ln\frac{\rho}{\rho_0}\alpha)_\eta - 2\xi f_\eta(\ln\frac{\rho}{\rho_0}\alpha)_\eta\right] - \frac{4\xi}{3\alpha\theta^*}\left(\frac{\dot{\sigma}_e t_s}{\rho RT_0} - \frac{5}{2}\theta\dot{\alpha}t_s\right) + 2\xi f_\eta\frac{\theta}{\theta^*}\frac{1}{\theta^*}\frac{d\theta^*}{d\xi}$$

$$\Gamma = b_3\alpha\left[\ln\frac{\rho}{\rho_0}(g+\theta)\right]_\eta,\ g=T_a/T_0,\ \theta=T_e/T_0$$

The symbols Pr, Le, and λ denote Prandtl number, Lewis number, and thermal conductivity, respectively, of heavy particles, λ_e denotes electron thermal conductivity, I ionization potential, R gas constant, T_0 room temperature, $\dot{\alpha}$ net degree of ionization rate, and $\dot{\sigma}_e$ electron energy production rate.

Both production rates $\dot{\alpha}$ and $\dot{\sigma}_e$ are given by the formulas evaluated by Hoffert and Lien[4] on the basis of the two-step ionization process which includes the first electronically excited state of argon. The transport parameters b_j ($j=1,2,3,4$) are given by the formulas evaluated by Jaffrin[5] on the basis of the hard sphere model and the mixture rule. Then, these parameters can be expressed as

$$b_1 = \frac{\rho}{\rho_0}(Q_{aa})_0\left\{\frac{1-\alpha}{(1-\alpha)Q_{aa}+\alpha Q_{ia}} + \frac{\alpha}{(1-\alpha)Q_{ia}+\alpha Q_{ei}}\right\}\sqrt{g}$$

$$Pr=2/3,\ b_3=\frac{6}{5}\frac{\rho}{\rho_0}\frac{g+\theta}{\sqrt{g}}\frac{(Q_{aa})_0}{Q_{ia}}$$

$$b_4=\frac{3}{2}\frac{\rho}{\rho_0}\alpha(Q_{aa})_0(m_a/m_e)^{1/2}\sqrt{\theta}\{\alpha(1+\sqrt{2})Q_{ei} + \sqrt{2}(1-\alpha)Q_{ea}\}^{-1}$$

where the quantity Q_{st} is the effective hard sphere cross section for particles s and t, and m_a/m_e the mass ratio of an atom and an electron.

The boundary conditions are given as

$$\eta=0{:}\ f=0,\ F_j=K_j,\ (j=1,2)\ (F_j)_\eta=K_jF_j,\ (j=3,4) \tag{5a}$$

$$\eta\to\infty{:}\ F_1=F_2=0,\ F_3=F_4=1 \tag{5b}$$

The coefficients K_j ($j=1,2,3,4$) can be given as

$$K_1=U_0/u^*-1,\ K_2=(RT_0/H_0)\left(\frac{5}{2}\theta+\frac{I}{RT_0}\right)\alpha$$

$$K_3=\frac{\rho}{\rho_0}\frac{\chi}{b_3}\left(\frac{6}{5}\xi\theta\right)^{1/2} - \frac{\rho_\eta}{\rho} - \frac{g_\eta+\theta_\eta}{g+\theta}$$

$$K_4=\alpha(2\xi\theta)^{1/2}(12/125)^{1/2}\frac{\rho}{\rho_0}\frac{\chi}{b_4}\left\{\frac{2+\Phi}{4}\left(\frac{8m_a}{\pi m_e}\right)^{1/2}\exp(-\Phi)-\frac{5}{2}\right\}$$

$$\chi=(\rho_0U_0^2t_s/\mu_0)^{1/2}/[U_0/(\tfrac{5}{3}RT_0)],\ \Phi=\phi/\phi_e$$

where ϕ is the sheath potential, and ϕ_e the electron temperature expressed in eV units. The boundary conditions at the wall are obtained from the no-slip condition, the room-temperature wall condition, and the continuity of the ion mass flux and electron energy flux at the sheath edge.[6,7] The sheath is assumed collision free.

Numerical Procedure

The conservation equations, Eqs. (4), with the boundary conditions Eqs. (5a) and (5b), are numerically solved by a finite-difference scheme. In actual calculations, we use the coordinates S and N defined by

$$S=1-[\ln(1+\xi)/\xi],\ \eta=BN/(1-N) \tag{6}$$

Substituting Eq. (6) into Eq. (4), we obtain

$$a_j'(F_j)_{NN}+b_j'(F_j)_N+c_j(F_j)_S=d_j\ \ (j=1,2,3,4) \tag{7}$$

With this transformation of variables, the semi-infinite plane with the conventional coordinates (ξ, η) is projected onto a unit quadrate with the reduced coordinates (S, N). The transformation of Eq. (6) has been used for the numerical solutions of the Navier-Stokes equations for a laminar incompressible flow past a parabolic cylinder[8] and for a semi-infinite flat plate with injection or suction.[9]

With these variables, the conservation equations, Eqs. (7), are numerically solved by a finite-difference scheme, which is based on the method of quasilinearization.[3] In the first step of iteration, the coefficients a_j', b_j', c_j and d_j are evaluated from an initial guess to the solutions for F_j, and regarded as known quantities. The wall boundary conditions, Eqs. (5a) and (5b), are also regarded as quasilinear relations with known values of K_j. Then, the finite-difference equations of Eqs. (7) can be solved by using matrices. The iteration process is continued till the sequence of linear solution converges to the solution of the nonlinear equations.

The N-derivatives of Eqs. (7) are replaced by three-point central differences, while the S-derivative is replaced by a two-point backward difference. The resultant difference equations of Eqs. (7) become simultaneous equations of tri-diagonal form for $(F_j)_{m,n}$ at the grid point (m,n), and can be solved by an elimination method.

Illustrative Example and Discussion

As an illustrative example, a frozen boundary-layer solution is shown for the case of shock Mach number $M_s=14$, initial pressure $p_0=1$ Torr, and initial gas temperature $T_0=300$ K. The initial degree of ionization immediately behind a shock front is assumed to be $\alpha^*=10^{-6}$. Calculations were made with 50 steps in the S-direction ($\Delta N=0.004$).

We may determine the values of the scaling factors t_s and B, such that the ionization relaxation region and the viscous layer occupy about half of the whole regions of the reduced coordinates ($0<S<0.5$, $0<N<0.5$). After some trial calculations, the values of the scaling factors were chosen as $t_s=40\times10^{-6}$ sec and $B=40$.

Figures 1 and 2 show profiles of the atom temperature and the electron temperature in the frozen boundary layer where the ionization production processes of charged species and

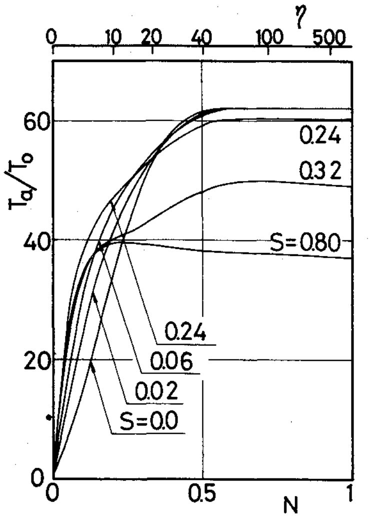

Fig. 1 Atom temperature profiles.

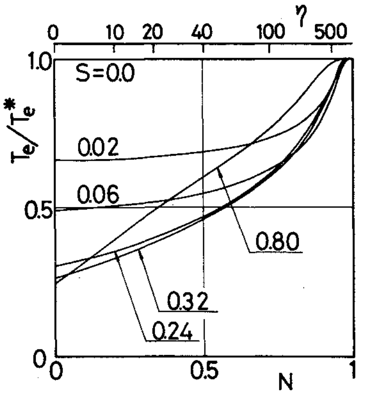

Fig. 2 Electron temperature profiles.

electron energy are ignored in the boundary layer. The two-layer structure of the flow is clearly seen. The thickness of the electron thermal layer is found to be $\eta = 500$ or above, while the thickness of the viscous boundary layer varies between $\eta = 10$ and $\eta = 40$. Thus, it is assured that the detailed structure of both the thin viscous boundary layer and the thick electron thermal layer can easily be obtained by using the reduced coordinates (S, N).

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Integral Momentum Equation for Flows with Entropy Gradients across Inviscid Streamlines

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Nomenclature

h	= enthalpy
H	= total enthalpy, $h + u^2/2$
p	= pressure
r	= normal distance from the body axis to the surface
s	= entropy
T	= temperature
u	= streamwise velocity
v	= velocity normal to the surface
x	= streamwise coordinate along the surface
y	= coordinate normal to the surface
δ^*	= displacement thickness, defined by Eq. (18)
δ^{**}	= displacement of inviscid streamlines by the boundary layer
ϵ	= geometry index for axisymmetric or two-dimensional flow
θ	= momentum thickness, defined by Eq. (17)
μ	= dynamic viscosity
ρ	= mass density
τ	= shear stress
ψ	= stream function, defined by Eq. (8)
Superscripts	
$'$	= fluctuating quantities
$-$	= mean quantities
$+$	= specific streamline outside of boundary layer
Subscripts	
e	= normal shock edge conditions
i	= inviscid
w	= surface
∞	= outside of boundary layer

Introduction

IN the flowfield around a blunt axisymmetric or two-dimensional body, the entropy of the fluid following a streamline of the inviscid flow is determined by upstream conditions and the local angle of the shock through which the streamline passes. Curved shocks therefore generate entropy gradients in the inviscid flow which, in turn, result in inviscid velocity gradients normal to the surface. The boundary-layer integral momentum equations and appropriate definitions of the momentum and displacement thicknesses valid for this situation are developed in this paper.

Inviscid Flowfield

The total enthalpy H is assumed to be constant throughout the inviscid flowfield, so that an energy balance yields

$$h_i + (u_i^2/2) = H_i \tag{1}$$

and

$$dh_i = -u_i \, du_i$$

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